A review of theoretical treatments of shock-tube attenuation

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A critical review is given of theories dealing with viscous effects in shock tubes for cases in which the boundary layer is thin compared with the diameter. In particular an examination of the radial variation of flow quantities is used to show that the approach of Mirels (1957) is based on firmer ground than that of Trimpi & Cohen (1955). It is also shown that the small perturbation theories of these and other authors lead to an equation for running time identical in form to that derived by Roshko (1960) from an asymptotic analysis of boundary-layer leakage through the interface.

1. Introduction

The performance of a real shock tube differs considerably from that predicted by the idealized theory which assumes inviscid flow and instantaneous diaphragm opening, and ignores such possibilities as combustion at the interface between driver and test gas, initial non-uniformity of the driver, and chemical kinetic effects. The effect of boundary-layer formation behind the shock has been studied by many authors, in two main ways:

(i) By introducing small perturbations representing the displacement effect of the boundary layer on the flow in the inviscid core (Trimpi & Cohen 1955, Mirels 1957, Demyanov 1957, Spence & Woods 1960); these treatments are concerned primarily with the motion of the shock wave.

(ii) By means of an asymptotic type of analysis aimed at predicting the ultimate running time from considerations of boundary-layer leakage through the interface (Duff 1959, Hooker 1961, Roshko 1960, Anderson 1959, Ackroyd 1963, Appleton & Musgrove 1963).

The linearized theories of type (i) are necessarily incomplete, since no entirely satisfactory way has been found of allowing for the boundary layer in the expanded driver. (For recent reviews of the *boundary-layer* part of the problem, see Stewartson 1960 and Becker 1961.) Moreover, to the authors' knowledge no mathematical discussion has been given of the mechanism by which the boundary layer influences the flow, previous arguments having been physical in character, leading to quite different formulations by Mirels and by Trimpi & Cohen. To clear up this point, an averaging procedure is applied to the axially symmetric flow in a circular tube, leading to one-dimensional equations of continuity, momentum and energy for suitably defined mean quantities, applied in §2.2. Inviscid flow in a quasi-cylindrical tube with small area changes, a closely related problem, had previously been considered by Chester (1954). The solution of the resulting equations, in linearized form, is then carried in §2.3 to the point of writing all flow variables in terms of the shock velocity variation, assuming this is known, and of a boundary-layer term.

These theories must, however, break down at large values of x/d (x = distance from diaphragm, d = diameter), both because the application of boundary conditions at the unperturbed shock position is by then unrealistic, and because the boundary layer well behind the shock becomes too thick. In §3 their relation to theories of type (ii) is explicitly displayed, and it is shown that if the shock velocity finally comes to a constant value, the equation for running time derived by Roshko (1960) is again obtained. From this follows Roshko's important conclusion that the running time approaches a constant value, confirming the experimental finding that beyond a certain stage increasing the length of a tube brings no advantage.

Superimposed on the attenuation due to viscous action are the departures from ideal shock-tube behaviour due to the other causes mentioned earlier. These are referred to much more briefly in the remainder of the paper.

2. Linearized attenuation theory

2.1. *Viscous effects*

The way in which boundary layers form behind the shock and expansion waves is indicated in figure 1, which also shows an (x, t)-diagram of the unperturbed flow. The effect of the boundary layers would be to perturb the traces of the shock and interface, shown here as straight lines. The first comprehensive treatments of shock attenuation due to this process were given by Trimpi & Cohen (1955) and Mirels (1957). These are one-dimensional small-perturbation theories, using the acoustic approximation. The quantity assumed small is the ratio of boundary-layer thickness (more precisely of the maximum displacement thickness) to tube radius, and perturbations of pressure, velocity and shock speed are all proportional to this. Both methods lead to computations based on characteristic networks, from which the shock path and all other flow perturbations can be constructed. (In fact, as will be pointed out in §2.3, the computation could be much simplified by solving a certain functional equation for the shock speed.)

In these treatments the mechanisms whereby boundary layers influence the core flow are introduced by means of physical arguments, without examination of the governing equations. Trimpi & Cohen introduce a wall-friction term in the momentum equation, and an entropy production term arising from wall heat transfer and viscous dissipation in the energy equation. Mirels, on the other hand, introduces only a mass addition term in the continuity equation (which is omitted from the Trimpi & Cohen analysis).

An integration across the tube of the governing equations for axially symmetric flow is given in the next paragraph. From this it is clear that appropriately-averaged quantities outside the boundary layer satisfy the one-dimensional energy and momentum equations without viscous terms, but that a displacement term appears in the continuity equation: in other words, it appears that Mirels's formulation is correct, and that of Trimpi & Cohen quite wrong. (If the boundary layer completely filled the tube, however, the physical arguments used by Trimpi & Cohen would have more force, but this state of affairs is not consistent with the perturbation analyses under review.)

The Mirels formulation was also used by Demyanov (1957), who obtain in closed form (in contrast to the elaborate construction method of Mirels) the solution for laminar boundary-layer growth between shock and interface. Demyanov's main object was to demonstrate that attenuation of the shock and

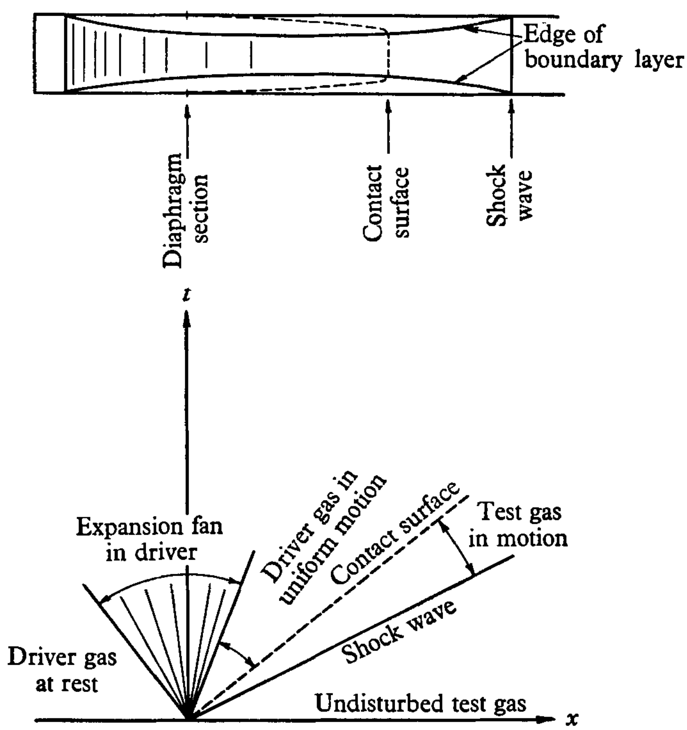

FIGURE 1. Schematic diagrams of flow in a shock tube.

acceleration of the interface are necessarily produced by this boundary layer. This work was extended by the present authors (1960) to cover turbulent boundary-layer growth, and to give expressions for pressure variation behind the shock. Predictions based on the theory, although of the right order, are not completely satisfactory; in particular the shock attenuation for a given nominal Mach number is predicted to be less with a lighter driver, whereas Jones (1957) found the opposite trend in tests using hydrogen and helium. One obvious omission from the theory lay in ignoring the boundary layer behind the interface, some account of which was taken by Mirels and by Trimpi & Cohen (although as remarked the second of these approaches appears unsound).

2.2. *Unsteady axially-symmetric flow in a constant-diameter tube*

With velocity components u, v in the axial (x) and radial (r) directions, the equations of continuity, momentum and energy are

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)+\frac{1}{r}\frac{\partial}{\partial r}(\rho v r) = 0, \tag{1}$$

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial r}\right)+\frac{\partial p}{\partial x} = \frac{1}{r}\frac{\partial}{\partial r}(\tau r), \tag{2}$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial r}\right)+\frac{\partial p}{\partial r} = 0, \tag{3}$$

$$\rho T\left(\frac{\partial S}{\partial t}+u\frac{\partial S}{\partial x}+v\frac{\partial S}{\partial r}\right) = \text{dissipation} + \text{heat-conduction terms}, \tag{4}$$

where τ is the shear stress (viscous + turbulent) and S is the specific entropy equal to $\ln(p/\rho^{\gamma})$ + const. In an ideal inviscid shock-tube flow we should have u, p, ρ and S constant, and $v = 0$. We may therefore assume (in the regime to which small-perturbation theory is applicable) that *outside the boundary layer* u, p, ρ and S depart from constant values by amounts of order δ/a, where δ is the local boundary-layer thickness and a the radius of the tube, while v is of order δ/a. Define a mean velocity

$$\overline{u}(x,t) = \frac{2}{(a-\delta)^2}\int_0^{a-\delta} ru\,(x,r,t)\,dr, \tag{5}$$

with similar definitions of mean pressure $\overline{p}$ and density $\overline{\rho}$, and write

$$u(x,r,t) = \overline{u}(x,t)+u'(x,r,t), \tag{6}$$

so that

$$\int_0^{a-\delta} ru'(x,r,t)\,dr = 0. \tag{7}$$

Then u', p', ρ' and v, being produced by boundary-layer effects, are all of order δ/a, as also are the departures of $\overline{u}$, $\overline{p}$, $\overline{\rho}$ from the values u_0, p_0, ρ_0 say which would hold in inviscid flow.

Multiplication of equation (1) by $2r/(a-\delta)^2$ and integration from 0 to $a-\delta$ gives

$$\frac{2}{(a-\delta)^2}\int_0^{a-\delta} r\frac{\partial \rho}{\partial t}dr+\frac{2}{(a-\delta)^2}\int_0^{a-\delta} r\frac{\partial}{\partial x}\{(\overline{\rho}+\rho')(\overline{u}+u')\}\,dr = -\frac{2\rho_\delta v_\delta}{a-\delta}, \tag{8}$$

where suffix δ denotes values at the edge of the boundary layer, namely $r = a-\delta$.

Now

$$\frac{\partial \overline{\rho}}{\partial t} = \frac{2}{(a-\delta)^2}\int_0^{a-\delta} r\frac{\partial \rho}{\partial t}dr+\frac{4}{(a-\delta)^3}\frac{\partial \delta}{\partial t}\int_0^{a-\delta} r\rho\,dr-\frac{2}{(a-\delta)^2}\frac{\partial \delta}{\partial t}(a-\delta)\rho_\delta$$

$$= \frac{2}{(a-\delta)^2}\int_0^{a-\delta} r\frac{\partial \rho}{\partial t}dr+\frac{2}{a-\delta}\frac{\partial \delta}{\partial t}(\overline{\rho}-\rho_\delta). \tag{9}$$

The second term on the right of (9) is $O(\delta/a)^2$, so by excluding terms of this order we can replace the first term on the left of (8) by $\partial\bar{\rho}/\partial t$. Likewise with use of (7) the second term on the left of (8) becomes

$$\frac{2}{(a-\delta)^2}\int_0^{a-\delta} r\frac{\partial}{\partial x}(\bar{\rho}\bar{u}+\rho' u')\,dr.$$

The $\rho' u'$ term makes a contribution of order $(\delta/a)^2$, which is again excluded, and by a similar argument to that leading to equation (9), the remaining term reduced to $\partial(\bar{\rho}\bar{u})/\partial x$ to the same order. Equation (8) can thus be rewritten

$$\frac{\partial\bar{\rho}}{\partial t}+\frac{\partial}{\partial x}(\bar{\rho}\bar{u}) = -\frac{2\rho_0 v_\delta}{a}, \tag{10}$$

excluding terms of order $(\delta/a)^2$.

When the same procedure is applied to the axial momentum equation (2), the term $v\,\partial u/\partial r = v\,\partial u'/\partial r$ is $O(\delta/a)^2$ so makes a negligible contribution, and from the remaining terms we obtain

$$\frac{2}{(a-\delta)^2}\int_0^{a-\delta}(\bar{\rho}+\rho')\left\{\frac{\partial}{\partial t}(\bar{u}+u')+(\bar{u}+u')\frac{\partial}{\partial x}(\bar{u}+u')\right\}r\,dr+\frac{\partial\bar{p}}{\partial x} = 0$$

(the pressure derivative being approximated in the same way as the density derivative (9)). Examination of the double and triple products easily shows that to order δ/a, this is just

$$\bar{\rho}\left(\frac{\partial\bar{u}}{\partial t}+\bar{u}\frac{\partial\bar{u}}{\partial x}\right)+\frac{\partial\bar{p}}{\partial x} = 0. \tag{11}$$

Likewise from the energy equation (4), in which the right-hand side is zero outside the boundary layer, we obtain

$$\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)\ln\left(\frac{\bar{p}}{\bar{\rho}^{\gamma}}\right) = 0. \tag{12}$$

Equations (10)–(12) (of which the last two are derived in more detail in the Appendix) can now be looked on as continuity, momentum and energy equations for a one-dimensional flow with a mass addition, represented by the right-hand side of (10), which must be found from a separate boundary-layer calculation. These are precisely the equations that were treated by Demyanov, although it is now clear that his interpretation was inconsistent, being based on the assumption that flow variables outside the boundary layer were independent of radial position.

The radial momentum equation (3) is now uncoupled from the system; its solution however would require knowledge of the shock curvature. Investigations of this are referred to in §5 below.

2.3. *Flow behind an attenuating shock*

Consider now the flow behind a shock wave which, but for viscous effects behind it, would travel at constant speed U_0 into gas at rest. Suppose the density, pressure and velocity behind the shock would then be ρ_0, p_0 and u_0 respectively and the speed of sound $c_0 = (\gamma p_0/\rho_0)^{\frac{1}{2}}$. The boundary-layer thickness behind the shock will be a function of U_0t-x only (varying as $(U_0t-x)^{\frac{1}{2}}$ or $(U_0t-x)^{\frac{4}{5}}$ for purely laminar or purely turbulent flow, and containing as a parameter a Reynolds number based on conditions behind the shock). If the perturbations due to boundary-layer action are small, we can write (10) and (11), with the aid of (12), in an acoustic approximation as

$$\frac{1}{c_0^2}\frac{\partial \overline{p}}{\partial t}+\frac{u_0}{c_0^2}\frac{\partial \overline{p}}{\partial x}+\rho_0\frac{\partial \overline{u}}{\partial x} = -\frac{2\rho_0 v_\delta}{a}, \tag{13}$$

$$\rho_0\left(\frac{\partial \overline{u}}{\partial t}+u_0\frac{\partial \overline{u}}{\partial x}\right)+\frac{1}{c_0^2}\frac{\partial \overline{p}}{\partial x} = 0. \tag{14}$$

If we set

$$v_\delta = \frac{u_0 a}{c_0}K'\left(\frac{U_0t-x}{U_0-u_0}\right) = \frac{u_0 a}{c_0}K'(\zeta) \tag{15}$$

say, where $K(\zeta)$ is a function which may be supposed known, either from a boundary-layer calculation or, for example, from heat-transfer measurements, the solution of (13) and (14) can be written

$$\frac{\overline{u}-u_0}{u_0} = f(\xi)+g(\eta)+\left(\frac{2M_0}{1-M_0^2}\right)K(\zeta), \tag{16}$$

$$\frac{\overline{p}-p_0}{\rho_0 u_0 c_0} = f(\xi)-g(\eta)+\left(\frac{2M_0^2}{1-M_0^2}\right)K(\zeta), \tag{17}$$

where $\xi = \dfrac{(c_0+u_0)t-x}{c_0+u_0-U_0}$, $\eta = \dfrac{(c_0-u_0)t+x}{c_0-u_0+U_0}$, $M_0 = \dfrac{U_0-u_0}{c_0}(<1)$.

If suffix s denotes conditions immediately behind the shock, whose instantaneous speed is U_s say, then for small perturbations in shock motion we can write

$$\overline{u}_s-u_0 = \lambda u_0(U_s-U_0)/U_0, \tag{18}$$

$$\overline{p}_s-p_0 = 2\mu\rho_0u_0(U_0-u_0)(U_s-U_0)/U_0, \tag{19}$$

where the derivatives λ, μ, which are easily computed from the Rankine–Hugoniot relations, have been defined in such a way that both tend to unity for a strong shock. If therefore we write the perturbation in shock speed at time t as

$$U_s-U_0 = U_0F(t) \tag{20}$$

and apply the boundary conditions (18) and (19) at the unperturbed shock position $x = U_0t$, where $\xi = \eta = t$, then since $K(0) = 0$ these become

$$f(t)+g(t) = \lambda F(t), \quad f(t)-g(t) = 2\mu M_0F(t)$$

and

$$f(t) = (\tfrac{1}{2}\lambda+\mu M_0)F(t), \quad g(t) = (\tfrac{1}{2}\lambda-\mu M_0)F(t). \tag{21}$$

Substitutions of these expressions in (16) and (17) therefore gives the variation of pressure and velocity behind the shock in terms of the shock speed and boundary-layer growth.

To go further and *calculate* the shock perturbation $F(t)$, it would be necessary in general to perform a further boundary-layer calculation for flow behind the interface. This would be very complicated and uncertain, although it has been attempted by Mirels. The calculation would lead to expressions for pressure and velocity perturbations in the expanded driver in the same functional form as (16) and (17), in the region between the interface and the leading characteristic of the expansion. By applying the condition that the expansion must begin from rest, it would then be possible to express one of the unknown functions corresponding to f and g in terms of the other. Thus free-stream conditions behind the interface would be expressed in terms of a single unknown function ϕ say, together with a boundary-layer function, J say, corresponding to $K(\zeta)$. Then by equating both pressure and velocity perturbations just behind the interface to those just in front, we should have two relations between the unknown functions ϕ and F, reducible by elimination of ϕ to a single functional equation expression $F(t)$ in terms of the known boundary-layer functions K and J. This is the basis of the analysis given by Spence & Woods (1960) for treating the case of combustion behind the interface.

If, however, we are prepared following Demyanov to omit the influence of the boundary layer behind the interface, the downstream boundary condition that the flow perturbations should be compatible with isentropic expansion of the driver from rest takes the much simpler form

$$(\bar{p}_c - p_0)/(\bar{u}_c - u_0) = -(c\rho)_3, \tag{22}$$

where suffix c denotes conditions just in front of the interface, c is the sound speed, and suffix 3 refers to the expanded driver. The boundary conditions at this point are applied on the line $x = u_0 t$, where $\xi = t/(1-M_0)$, $\eta = t/(1+M_0)$, $\zeta = t$. When these values are inserted in (16) and (17), (22) becomes, with the use of (21),

$$AF\left(\frac{t}{1-M_0}\right) + BF\left(\frac{t}{1+M_0}\right) + CK(t) = 0, \tag{23}$$

where $\left.\begin{matrix} A \\ B \end{matrix}\right\} = \left(1 \pm \frac{\rho_0 c_0}{\rho_3 c_3}\right)(\tfrac{1}{2}\lambda \pm \mu M_0), \quad C = \left(\frac{2M_0}{1-M_0^2}\right)\left(1 + \frac{\rho_0 c_0}{\rho_3 c_3} M_0\right).$

(23) is a functional equation giving the shock perturbation function $F(t)$ in terms of the boundary-layer growth function $K(t)$. The solution can be written down as

$$F(t) = -\frac{C}{A} \sum_{n=0}^{\infty} \left(-\frac{B}{A}\right)^n K\left\{\frac{(1-M_0)^{n+1} t}{(1+M_0)^n}\right\}. \tag{24}$$

This solution is equivalent to the construction method of Mirels. If the boundary layer is purely laminar or purely turbulent K is of the form $K_1 t^m$ with $m = \frac{1}{2}$ or $\frac{4}{5}$ in the two cases. (24) then reduces to

$$F(t) = -C\left[\frac{A}{(1-M_0)^m} + \frac{B}{(1+M_0)^m}\right]^{-1} K_1 t^m. \tag{25}$$

This is the 'self-similar' solution derived *ab initio* by Mirels & Braun (1962). In the more general case referred to when the boundary layer behind the interface is taken properly into account, the functional equation satisfied by $F(t)$ could be put in the form

$$F(t) + AF(\alpha t) + BF(\beta t) = L(t) \tag{26}$$

say, where $1 > \alpha > \beta$, A and B being different from the constants in (23), and $L(t)$ being a known boundary-layer disturbing function. The solution for this equation can also be written down; it is

$$F(t) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} (-1)^n \binom{n}{m} A^{n-m} B^m L(\alpha^{n-m}\beta^m t) \tag{27}$$

as was first noted by Gunderson (1958) in a discussion of one-dimensional flows with weak entropy addition.

3. Running time

The linear theory just described predicts deceleration of the shock and acceleration of the interface, so that the theoretical running time grows to a maximum and then decreases to zero, in contrast to the most commonly observed state of affairs in which the running time reaches a maximum then stays more or less constant, while, once this stage has been reached, the shock speed decreases more slowly.

Linearized theory must clearly have become invalid well before it predicts the shock being overtaken by the interface, if only because the lines on which the boundary conditions are applied are by then far from the true positions of shock and interface. The failure of the Demyanov-type theories may also be due in part to neglect of the boundary layer behind the interface.

To get at the running time directly a number of writers, following up a qualitative explanation due to Duff (1959), have used an asymptotic approach based on the fact that the mass of fluid between the shock and the 'interface' (i.e. the contact surface in the inviscid core) increases at a rate equal to the difference between that at which the shock passes through new fluid, and that at which boundary-layer leakage through the interface takes place. Assuming both shock speed, and density between shock and interface, constant, calculation of these rates then gives the running time at each station. The most complete discussions along these lines are those of Roshko (1960) and Hooker (1961). We show here how the equation studied by Roshko can be derived from the analysis of the last section.

The distance l say between shock and interface at a given time grows according to the equation

$$dl/dt = U_s(t) - u_c(t), \tag{28}$$

where U_s, u_c are speeds of shock and interface respectively. Roshko's assumption is effectively

$$F(t) = \text{const.} = F_\infty \tag{29}$$

say, where $F(t)$ is the shock perturbation function defined by (20). Consistently with this we must also take $f[t/(1 \pm M_0)]$, which give the perturbations at the interface, as being $(\frac{1}{2}\lambda \pm \mu M_0)\, F(\infty)$, and substitution of (16) and (20) in (28) then gives

$$dl/dt = W_0 - W_1 K(\zeta_c) \tag{30}$$

say, where

$$W_0 = U_0(1+F_\infty) - u_0(1+\lambda F_\infty), \quad W_1 = 2M_0 u_0/(1-M_0^2),$$

and ζ_c denotes the value of ζ at the interface. As noted above, the shock and interface positions are by now well removed from the lines $x = U_0 t$, $x = u_0 t$, so it no longer gives a good approximation to take $\zeta_c = t$ as in the previous section; instead we must take $\zeta_c = l/(U_0 - u_0)$, when (30) takes the form

$$dl/dt = W_0 - W_2(U_0 - u)^{-m} l^m, \tag{31}$$

with $m = \frac{1}{2}$ or $\frac{4}{5}$ for laminar (the case treated by Roshko) or turbulent boundary layers. The initial condition is $l = 0$ when $t = 0$. The solution of this equation is straightforward and shows $l(t) \to$ a constant as $t \to \infty$. Since the variation of density between shock and contact surface has been implicitly taken into account in the present derivation but not in Roshko's, there may be slight numerical differences between the two.

4. The finite opening time of the diaphragm

White (1958) has pointed out that anomalously fast shocks sometimes observed at very high diaphragm pressure ratios could be caused by delay in their formation due to the finite opening time of the diaphragm. The only observable predicted by his theory is a transient maximum shock speed, greater than that given by ideal shock-tube theory. Agreement between his theoretical and experimental results is remarkably good, but he states that this is probably partially fortuitous. Kireyev (1962) has made more detailed calculations based on a plausible model of diaphragm opening, in which the trajectory of the accelerating shock and the non-uniform flow behind it are computed. His results agree well with those of White.

Kireyev finds that in a high-pressure shock tube, the inertia of the diaphragm petals dominates elastic/plastic bending forces in resisting opening, and this fact enables a scaling law for diaphragm opening to be deduced, namely that for geometrically similar diaphragms rupturing in the same manner, opening times will be proportional to $(\rho_D d/p)^{\frac{1}{2}}$, where ρ_D is the mass per unit area of the diaphragm, p the pressure difference across it, and d a characteristic dimension of the tube cross-section.

In relating this to the subsequent flow, the properties of the test and driver gas have to be brought in, in a rather complicated manner, and no universal scaling law predicting, say, shock-formation distance in terms of all the relevant parameters seems possible. Nevertheless, there appear to be two general trends, indicated respectively in figures 2 and 3 (which are derived from figures 8 and 9 of White's paper): (1) Shock-formation distance increases steadily with Mach number. (ii) For shocks at a given Mach number driven by different driver gases, the formation distance is shorter when the speed of sound in the driver is higher. (In fact the test gases in the curves of figure 3 are also different, but this effect seems less important.)

Gaydon (1963) quotes a shock-formation distance of about seven diameters. This strictly speaking is the distance for the compression waves following diaphragm rupture to coalesce into a plane shock front, but as can be seen from figure 2 the influence of the formation process on the motion of the shock persists for some considerably greater length.

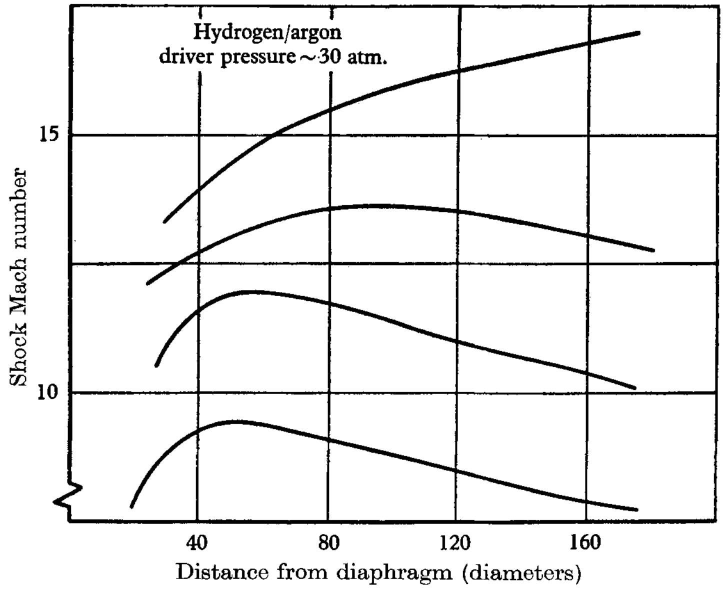

FIGURE 2. Shock acceleration due to finite diaphragm opening time for several pressure ratios (from White 1958, figure 8).

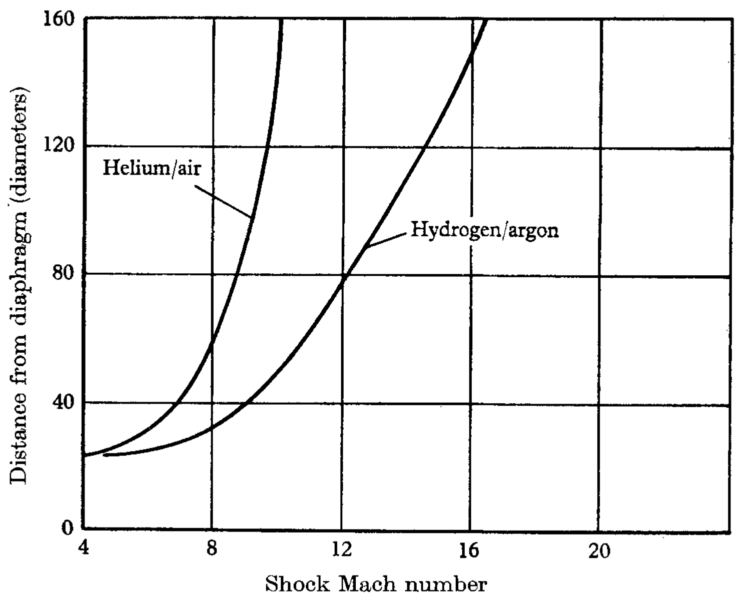

FIGURE 3. Shock formation distance *vs* Mach number with different drivers (from White 1958, figure 9).

White also gave a rough qualitative account of how mixing in the contact region, which would certainly accompany real diaphragm opening, would affect the flow. The method used by Spence & Woods (1960) to treat combustion could be applied equally to mixing, but in the absence of an adequate description of the mixing process it too would yield only qualitative results.

5. Other factors

5.1. *Initial non-uniformity of driver and test gas*

Non-uniform conditions in the driver at the instant of diaphragm rupture may affect the flow in a combustion-driven shock tube. Some possible models of this have been computed by Bird (1957). A related effect has been found in work at Cornell Aeronautical Laboratory in a tube driven by gas heated by electrical resistance elements. Here the whole high-pressure section becomes hot, and heat conduction down the tube causes a lengthwise temperature gradient in the test gas, leading to variation in the shock speed and non-uniformity in the flow.

5.2. *Combustion in the contact region*

Spence & Woods (1960) treated this using an idealized model of combustion, and explained qualitatively some observed features of flows in hydrogen/air shock tubes.

5.3. *Chemical relaxation effects*

The one-dimensional flow of a reacting gas behind a shock has been treated by many writers, and measurements of its non-uniformity used to deduce the rates of the reactions involved. Spence (1961) has treated the piston problem, and shown that relaxation leads to attenuation; the shock initially travels at a speed appropriate to frozen conditions, but slows down to that appropriate to equilibrium as time increases. It is quite simple by combining this analysis with that of Spence & Woods to show that the same behaviour is to be expected following diaphragm burst in an inviscid shock-tube flow.

5.4. *Shock curvature*

Boundary-layer growth in the shock tube causes the shock wave to become curved. This effect is mentioned in a report on low-density shock tube work by Lin & Fyfe (1961) who point out its relevance to measurements made across the full diameter of the tube. It has also been studied experimentally by Duff & Young (1961) and theoretically by Hartunian (1961), Sichel (1962), and de Boer (1963). Hartunian's analysis is invalid near the wall, where a shock thickness is of the same order of magnitude as viscous layer thickness, and the boundary-layer approximation is inapplicable. Sichel's treatment is consistent in this region but limited to weak shocks. Neither can be compared with experiment, since both apply only to plane flows. In de Boer's study Hartunian's theory is extended to rectangular and circular tubes.

6. Non-uniformity behind the reflected shock

In a study of the interaction of the reflected shock with the shock-tube boundary layer, Mark (1958) observed that separation greatly disturbed the flow behind the shock. However, separation is not necessary for non-uniformity to occur. Rudinger (1961) has shown that the non-uniformities behind the attenuating incident shock are transmitted through the reflected shock, causing the flow behind it to be unsteady. The same problem has been treated theoretically and experimentally by Kamimoto, Akamatsu & Hasegawa (1962), and theoretically by Woods (1962), who also discusses the relation between his work and that of Rudinger.

Strehlow & Case (1961) have however reported measurements of density variation behind the reflected shock which do not agree with calculations made by Rudinger's method.

7. Conclusions

An examination of the radial variations of flow quantities in a constant-diameter shock tube in which the boundary layer is thin compared with the diameter has shown that Mirels's formulation of the linearized equations for an attentuation theory, in which a small perturbing term (due to boundary-layer growth) appears in the continuity equation, is the correct one for this case.

The effects of viscosity and of finite diaphragm-opening time will always be present, and if one tries to reduce viscous attenuation by reducing the length-to-diameter ratio of the tube, the analysis of Kireyev shows that the relative effect of finite diaphragm opening time will be greater.

Since the effects of these factors cannot be computed in advance, it seems necessary to monitor the shock trajectory and at least one other flow variable—e.g. heat transfer or pressure variation—in order to build up a fairly complete picture of flow conditions in a tube. An example of how this can be done occurs in the recent work by Holbeche & Spence (1964), in which directly-measured temperatures behind an attenuating shock wave have been related to measurements of shock speed and pressure.

We are grateful to Dr D. R. White for permission to base figures 2 and 3 on curves in his paper (*loc. cit.*).

Appendix

Derivation of equations (11) *and* (12), *for quantities averaged across the inviscid core*

The one-dimensional equations of motion (11) and (12) are derived from the corresponding complete equation (2) and (4) in the same way as (10) was obtained from (1), in the main text.

When $\bar{u}+u'$, $\bar{p}+p'$, $\bar{\rho}+\rho'$ are substituted for u, p and ρ in (2) (according to the definition of (6) and (7)), multiplication of that equation by $2r/(a-\delta)^2$ and integration with respect to r from 0 to $a-\delta$ give

$$\frac{2}{(a-\delta)^2}\int_0^{a-\delta}(\bar{\rho}-\rho')\frac{\partial(\bar{u}+u')}{\partial t}r\,dr+\frac{2}{(a-\delta)^2}\int_0^{a-\delta}(\bar{\rho}+\rho')(\bar{u}+u')\frac{\partial(\bar{u}+u')}{\partial x}r\,dr$$
$$+\frac{2}{(a-\delta)^2}\int_0^{a-\delta}v\frac{\partial u'}{\partial r}r\,dr+\frac{2}{(a-\delta)^2}\int_0^{a-\delta}\frac{\partial}{\partial x}(\bar{p}+p')\,r\,dr=\frac{1}{(a-\delta)^2}\tau\bigg|_{r=a-\delta}. \quad (32)$$

From the definition of the barred and primed quantities, in expressions such as

$$\int_0^{a-\delta} \bar{\rho}\frac{\partial u'}{\partial t} r\,dr = \bar{\rho}\frac{\partial}{\partial t}\int_0^{a-\delta} u' r\,dr + \bar{\rho}(a-\delta)\frac{\partial \delta}{\partial t} u'\Big|_{r=a-\delta},$$

$$\int_0^{a-\delta} \bar{\rho}\bar{u}\frac{\partial u'}{\partial x} r\,dr = \bar{\rho}\bar{u}\frac{\partial}{\partial x}\int_0^{a-\delta} u' r\,dr + \bar{\rho}\bar{u}(a-\delta)\frac{\partial \delta}{\partial x} u'\Big|_{r=a-\delta},$$

and
$$\int_0^{a-\delta} \frac{\partial p'}{\partial x} r\,dr = \frac{\partial}{\partial x}\int_0^{a-\delta} p' r\,dr + (a-\delta)\frac{\partial \delta}{\partial x} p'\Big|_{r=a-\delta},$$

the first term on the right-hand side vanishes, and the second is of order $(\delta/a)^2$; terms containing products of two (or more) primed quantities are also of order $(\delta/a)^2$; and, by definition, $\tau|_{r=a-\delta}$ is vanishingly small. Equation (32) then reduces to

$$\bar{\rho}\left(\frac{\partial \bar{u}}{\partial t} + \bar{u}\frac{\partial \bar{u}}{\partial x}\right) + \frac{\partial \bar{p}}{\partial x} = 0,$$

which is equation (11).

Equation (4) written out fully is

$$\frac{\partial S}{\partial t} + u\frac{\partial S}{\partial x} + v\frac{\partial S}{\partial r} = (\rho T)^{-1}\left\{\mu\left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\mu}{Pr} r\frac{\partial h}{\partial r}\right)\right\}. \tag{33}$$

The right-hand side which represents the production of entropy due to viscous dissipation and heat conduction, may (if it is further assumed that the Prandtl number is nearly 1), be ignored outside the boundary layer. Put

$$S = \bar{S}(x,t) + S'(x,r,t), \tag{34}$$

where
$$\int_0^{a-\delta} S' r\,dr = 0$$

and S' is of order (δ/a). Then when the left-hand side of (33) is multiplied by $2r/(a-\delta)^2$ and integrated from 0 to $(a-\delta)$, there results

$$\frac{2}{(a-\delta)^2}\int_0^{a-\delta}\frac{\partial}{\partial t}(\bar{S}+S')\,r\,dr + \frac{2}{(a-\delta)^2}\int_0^{a-\delta}(\bar{u}+u')\frac{\partial}{\partial x}(\bar{S}+S')\,r\,dr$$
$$= -\frac{2}{(a-\delta)^2}\int_0^{a-\delta} v\frac{\partial S'}{\partial r} r\,dr$$
$$= O\{(\delta/a)^2\}.$$

It can be shown, as above, that

$$\int_0^{a-\delta}\frac{\partial S'}{\partial t} r\,dr = \frac{\partial}{\partial t}\int_0^{a-\delta} S' r\,dr + (a-\delta)\frac{\partial \delta}{\partial t} S'\Big|_{r=a-\delta} = O\{(\delta/a)^2\},$$

$$\int_0^{a-\delta}\bar{u}\frac{\partial S'}{\partial x} r\,dr = \bar{u}\frac{\partial}{\partial x}\int_0^{a-\delta} S' r\,dr + \bar{u}(a-\delta)\frac{\partial \delta}{\partial x} S'\Big|_{r=a-\delta} = O\{(\delta/a)^2\},$$

and
$$\int_0^{a-\delta} u'\frac{\partial \bar{S}}{\partial x} r\,dr = 0;$$

equation (33) reduces to

$$\partial\bar{S}/\partial t + \bar{u}\,\partial\bar{S}/\partial x = 0. \tag{35}$$

It remains to identify $\bar{S}$ with $\ln(\bar{p}/\bar{\rho}^{\gamma})$. Quite generally,

$$S/c_v = \ln p - \gamma \ln \rho + \text{const.},$$

or
$$(\bar{S}+S')/c_v = \ln(\bar{p}-p') - \gamma\ln(\bar{\rho}+\rho') + \text{const.}$$
$$= \ln\bar{p} - \gamma\ln\bar{\rho} + p'/\bar{p} - \gamma\rho'/\bar{\rho} + \text{const.} + O\{(\delta/a)^2\}.$$

Multiply both sides by $2r/(a-\delta)^2$ and integrate with respect to r from 0 to $a-\delta$; there results

$$\bar{S}/c_v = \ln \bar{p} - \gamma \ln \bar{\rho} + \text{const.},$$

so that (35) is equivalent to

$$\partial \ln (\bar{p}/\bar{\rho}^\gamma)/\partial t + \bar{u}\, \partial \ln (\bar{p}/\bar{\rho}^\gamma)/\partial x = 0,$$

which is equation (12).

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